J. Appl. Maths Mechs Vol. 57, No. 3, pp. 563–564, 1993
Printed in Great Britain.

0021–8928/93 $24.00+.00

THE METHOD OF CONTOUR DYNAMICS FOR THE EQUATIONS OF FREE CONVECTION†

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(*Received* 27 *May* 1992)

A class of spatial flows which obey the equations of free ideal convection in the Boussinesq approximation is presented. The method of contour dynamics is applied to these flows which enables one to reduce a problem to a one-dimensional one. Time laws are obtained for the change in the three principal quantities: the vertical vorticity, the horizontal divergence and the temperature deviation.

THE EQUATIONS of free convection in the Boussinesq approximation [1]

$$D_t\mathbf{v} = -\rho_0^{-1}\nabla p' - g\beta T' + \nu\Delta\mathbf{v}, \quad D_t = \partial_t + (\mathbf{v}\cdot\nabla)$$
$$\operatorname{div}\mathbf{v} = 0, \quad D_t T' = k\Delta T' \tag{1}$$

where $\mathbf{v} = (v_1, v_2, v_3)$ is the velocity, ρ_0 is the mean density, p' and T' are the deviations of the pressure and temperature, g is the acceleration due to gravity, ν is the kinematic viscosity, and k and β are the thermal diffusivity and the coefficient of thermal expansion, have solutions of the form

$$\mathbf{v} = (v_1(x,y,t), v_2(x,y,t), -z\sigma), \quad T' = zT'(x,y,t) \tag{2}$$

Here $\sigma = \partial_x v_1 + \partial_y v_2$ is the divergence in the horizontal plane in terms of which both the vertical component of the velocity and the horizontal components of the vorticity are expressed. In this case, the vortex field has the form

$$\boldsymbol{\omega} = (-z\partial_y\sigma, z\partial_x\sigma, \omega_3) \tag{3}$$

Hence, system (1) reduces to a system of equations for the three controlling quantities σ, T' and ω_3

$$D_t\sigma = \sigma^2 - g\beta T' + p_0 + \nu\Delta\sigma$$
$$D_t T' = \sigma T' + k\Delta T', \quad D_t\omega_3 = -\sigma\omega_3 + \nu\Delta\omega_3 \tag{4}$$

The quantity p_0 is associated with the pressure distribution. The equation of continuity is satisfied on account of the fact that the velocity field is specified in the form of (2).

In the case of ideal convection, when $\nu = 0$ and $k = 0$, the resulting system of equations is solved exactly in the case of piecewise-homogeneous distributions of the quantities σ, T' and ω_3. Actually, the general case of localized perturbations of this type reduces to the simplest form when σ, T' and ω_3 are concentrated in a domain D

$$\sigma(x,y,0) = \sigma_0\chi(D), \quad T'(x,y,0) = T_0'\chi(D), \quad \omega_3(x,y,0) = \omega_3^0\chi(D) \tag{5}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 160–162, 1993.

where $\chi(D)$ is a characteristic function of the domain $D=D(t)$. From (4) we obtain the system of ordinary differential equations

$$\sigma=\sigma^2-g\beta T'+p_0,\quad T'=\sigma T',\quad \dot{\omega}_3=-\sigma\omega_3 \tag{6}$$

since all the fluid particles in domain D are equivalent and Eqs (4) not only describe the Lagrangian dynamics of the individual fluid particles but also the global evolution of the fluid volume $D(t)$. In the case of a bounded domain D, the quantity $p_0=0$, since the pressure at infinity is independent of z. On solving the system of Eqs (6) when $p_0=0$, we obtain

$$\sigma(t)=(\sigma_0-g\beta T_0')/Q(t),\quad T'(t)=T_0'/Q(t)$$
$$\omega_3(t)=\omega_3^0 Q(t),\quad Q(t)=1-\sigma_0 t+\frac{1}{2}g\beta T_0' t^2 \tag{7}$$

Time laws [7] enable one to use the method of contour dynamics [3, 4] and, in this case, the equation describing the dynamics of the boundary contour ∂D preserves its form

$$\zeta_0=-\frac{\omega_3(t)+i\sigma(t)}{2\pi}\oint_{\partial D}\ln|\zeta-\zeta_0|d\zeta \tag{8}$$

Here $\zeta=x+iy$ are complex variables in the horizontal plane and the points ξ and ζ_0 belong to the boundary contour ∂D.

The time dependences (7) also enable one to draw a number of general conclusions concerned with the development of perturbations which are localized in the x, y plane, the approximate behaviour of which, ignoring the change in the shape of the domain $D(t)$, is determined by the signs of the quantities σ_0 and T_0' and the determinant $d=\sigma_0^2-2g\beta T_0'$.

In the case when $d<0$, we have $T_0'>\sigma_0^2/(2g\beta)>0$, that is, the perturbation has a warm (light) nucleus which, as follows from (7), contracts into a vortex filament since $\sigma(t)\to 0$, $T'(t)\to 0_+$ and $\omega_3(t)\to\infty$ when $t\to\infty$. In this case, the divergence $\sigma_0>0$ can be specified at the initial instant $t=0$ and the equality $\sigma=0$ holds when $t=t_0=\sigma_0/(g\beta T_0')$ and $\sigma(t)$ subsequently becomes negative, that is, divergence is replaced by convergence.

If $d\geqslant 0$, then $T_0'<\sigma_0^2/(2g\beta)$ and we have three regimes depending on the signs of T_0' and σ_0. When $T_0'>0$, $\sigma_0<0$, there is a contraction of the perturbation into a vortex filament. When $T_0'>0$, at the initial instant, we have divergence leading to an "eruption" $\sigma(t)\to+\infty$, $T'(t)\to+\infty$, $\omega_3(t)\to 0$ when $t\to t_1=t_0-\sqrt{(d)}/(g\beta T_0')$. And, finally, when $T_0'<0$, $\sigma(T)$ can take values with different signs and the time varies over the range $t_2<t<t_2=t_0+\sqrt{(d)}/(g\beta T_0')$. When $t\to t_{1+}$, we have $\sigma(t)\to-\infty$, $\omega_3(t)\to 0$, that is, the limiting possible initial state is a confluent filament with a cold nucleus which bursts since $\sigma(t)\to+\infty$, and $T'(t)\to-\infty$ when $t\to t_{2-}$. It should be noted that there is a spontaneous growth in the vorticity here which attains its maximum value when $t=t_0$ and, then $\sigma=0$ and T' also takes its maximum value.

Hence, under certain conditions (when $d<0$, $T_0'>0$), and when $d>0$, $\sigma_0<0$, $T_0'>0$), there is a contraction of the initial perturbation with a simultaneous intensification of the vorticity. Such processes are observed during the formation of intensive convective vortices and, then, the flows ascending along the axes of the vortices generate convergence, that is a converging radial motion of the medium close to the base of a vortex. By using the method of contour dynamics with appropriate initial data, it is possible to simulate unsteady processes of the formation of concentrated convective vortices.

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Translated by E.L.S.